Geometric quantization of the multidimensional Kepler problem

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Abstract. *The geometric quantization scheme of Czyz and Hess is applied to the $(n-1)$-dimensional quadric in complex projective space. As the quadric is the orbit manifold of the n-dimensional Kepler problem and the geodesic flow on the n-dimensional euclidean sphere, we thus obtain the quantum energy levels and their multiplicities for these Hamiltonian systems.*

1. INTRODUCTION

The n-dimensional Kepler problem in the Hamiltonian system (M, ω, H) where $M = (R^n \setminus \{0\}) \times R^n$ with coordinates $q_1 \dots q_n, p_1 \dots p_n$ and

$$(1) \qquad \omega = \mathrm{d}p \wedge \mathrm{d}q; \qquad H = \frac{1}{2}|p|^2 - \frac{1}{|q|}.$$

Using stereographic projection, Moser [1] has shown the equivalence of the regularized problem (1) with the geodesic flow on the sphere $S^n = \{\xi \in R^{n+1}; |\xi| = 1\}$ (see also [2] ch. IV §6). The geodesic flow on S^n is the Hamiltonian system

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(P, σ, G) where $P = \{\xi, \eta \in R^{n+1}; |\xi| = 1, \langle\xi, \eta\rangle = 0\}$ and

$$\sigma = d\xi \wedge d\eta; \qquad G = \frac{1}{2}|\xi|^2|\eta|^2 \tag{2}$$

where we consider σ and G restricted to P.

The orbits of problem (2) are the geodesics on S^n, i.e. the great circles. Those orbits which lie on a fixed energy hypersurface $G = \epsilon$ (fixed velocity), are parametrized by the points of the grassmanian of oriented 2-planes in R^{n+1}. This Grassmanian is the compact Hermitian symmetric space $SO(n+1)/SO(n-1) \times \times SO(2)$, which is known ([3] ch. XI) to be isometric to the nonsingular $(n-1)$-dimensional complex quadric Q_{n-1} with the natural Kahler metric.

$$Q_{n-1} = \left\{(z_1, \ldots, z_{n+1}) \in CP^n : \sum_{j=1}^{n+1} z_j^2 = 0\right\}. \tag{3}$$

In the present paper we apply the modified geometric quantization scheme of Czyz [4] and Hess [5] to the Kepler manifold Q_{n-1} for $n > 3$. In Sect. 3 we use algebro-geometric methods to determine quantum line bundles on Q_{n-1} and compute the dimensions of their spaces of holomorphic sections. This allows us to prove:

THEOREM 1. *The energy spectrum of the n-dimensional Kepler problem is*

$$E_N = -\frac{1}{2}(N + (n-3)/2)^{-2} \tag{4}$$

with corresponding multiplicities

$$\frac{2N + n - 3}{n - 1}\binom{N + n - 3}{N - 1} \tag{5}$$

where $N = 1, 2, \ldots,$.

This is an immediate consequence of

THEOREM 2. *The energy spectrum of the geodesic flow on* S^n *is*

$$\epsilon_N = \frac{1}{2}(N + (n-3)/2)^2 \tag{6}$$

with corresponding multiplicities

$$\frac{2N+n-3}{n-1}\binom{N+n-3}{N-1} \tag{7}$$

where $N = 1, 2, \ldots,$.

For the proofs see Sect. 4.

Remark 1. In the case when $n = 3$ the Hamiltonian system (1) was quantized geometrically by Simms [6], [7], who applied the Kostant - Souriau theory to Q_2. He computed the multiplicities via the Riemann - Roch - Hirzebruch theorem for complex surfaces. In Sect. 3 we use for this purpose simpler algebro-geometric devices, which allow us to avoid the computational problems arising in the Riemann - Roch - Hirzebruch formula for higer dimensions.

Remark 2. The quasiclassical energy spectrum and multiplicities for the problem (2) were discussed by Weinstein [8]. Later Ii [9] calculated the exact quasiclassical energy levels and multiplicities for (2), which agree exactly with the values we have obtained by geometric quantization.

Remark 3. Curiously, the technical details of our «cohomological» computation of the multiplicities in Sect. 3 are quite parallel to the «classical» computation of the multiplicities of the classical energy levels, i.e. with the standard computation of the dimensions of the spaces of homogeneous harmonic polynomials of a given degree in R^{n+1} (see e.g. [10] ch. III).

Remark 4. The reduction to the «orbit manifold» Q_{n-1} amounts in quantum--mechanical terms to transition from the Schrodinger to the Heisenberg picture [11]. Rawnsley [12] discusses geometric quantization of problem (1) in terms of a Kahler structure on the space T^*S^n.

Remark 5. Obviously perturbations destroy the high symmetry of the problem. In particular we do not have orbit manifolds. Thus it seems that this method does not apply even to complitely integrable cases as the geodesic flow on an elipsoid.

2. PRELIMINARIES

We consider hypersurfaces of constant energy ϵ on the phase space P.

$$P_\epsilon = \{(\xi, \eta) \in P : G(\xi, \eta) = \epsilon\}. \tag{8}$$

Denote by $\sigma_{\frac{1}{2}}$ the restriction of the symplectic form σ to the submanifold $P_{\frac{1}{2}}$.

The hypersurface

$$P_{\frac{1}{2}} = \{(\xi, \eta) \epsilon R^{n+1} \times R^{n+1}; |\xi| = |\eta| = 1, \langle \xi, \eta \rangle = 0\} = \\ = SO(n+1)/SO(n-1)$$

is the Stiefel manifold of oriented orthonormal 2-frames in R^{n+1}. The orbits of the geodesic flow on $P_{\frac{1}{2}}$ are exactly the orbits of the action of $SO(2)$ given by

$$(\xi, \eta) \longrightarrow (\xi \operatorname{cost} + \eta \operatorname{sint}, -\xi \operatorname{sint} + \eta \operatorname{cost}) \quad \text{for} \quad \begin{pmatrix} \operatorname{cost} & \operatorname{sint} \\ -\operatorname{sint} & \operatorname{cost} \end{pmatrix} \epsilon SO(2)$$

Thus the factor space of this action (the orbit manifold) is

$$SO(n+1)/SO(n-1) \times SO(2) = Q_{n-1}.$$

We denote the natural projection by $\Pi : P_{1/2} \to Q_{n-1}$. The projection $\Pi_\epsilon : P_\epsilon \to Q_{n-1}$ is obtained from Π by scaling the η component; as P_ϵ is obtained from $P_{\frac{1}{2}}$ by

(9) $$\xi \longrightarrow \xi, \qquad \eta \longrightarrow \sqrt{2\epsilon}\,\eta.$$

We scale the invariant Kahler metric g on Q_{n-1} so that the corresponding Kahler form Ω (see Sect. 3) becomes the invariant representative of the first Chern class of the hyperplane section bundle on Q_{n-1}. From the construction of the metric g ([3] ch. XI ex. 10.6) it is obvious that the forms $\Pi^*\Omega$ and $\sigma_{\frac{1}{2}}$ on $P_{\frac{1}{2}}$ differ by a constant factor. We assume (by an appropriate choice of «physical» constants) that

(10) $$\Pi^*\Omega = \sigma_{\frac{1}{2}}.$$

Formulas (9) and (10) imply

(11) $$\sigma_\epsilon = \Pi_\epsilon^*(\sqrt{2\epsilon}\,\Omega) = \Pi_\epsilon^*(\Omega_\epsilon).$$

Moser [1] has introduced a symplectomorphism between the phase spaces (M, ω) and (P, σ), which maps an energy hypersyrface $M_E = \{(p, q) \epsilon M; H(p, q) = = E\}$ onto a hypersurface P_ϵ for

(12) $$E = -\frac{1}{4\epsilon}$$

(see also [2] ch. IV 6).

We summarise the elements of the modified geometric quantization scheme of

Czyz [4] and Hess [5] which will be used in our proof.

Let X be a compact Kahler manifold with Kahler form h. By definition, a quantum line bundle L on (X, h) is a holomorphic line bundle L, whose first Chern class $c_1(L)$ satisfies

$$(13) \qquad c_1(L) = [h] - c_1(X)/2.$$

The corresponding quantum Hilbert space is (as a linear space), the space $H^0(X, L)$ of holomorphic sections of L. In our case (X, h) will be $(Q_{n-1}, \Omega_\epsilon)$. The existence of a quantum line bundle is obviously a condition on ϵ, which determines the energy spectrum. The dimensions of the corresponding quantum Hilbert spaces are the multiplicities.

3. HOLOMORPHIC LINE BUNDLES ON THE QUADRIC

On the n-dimensional complex projective space CP^n we have the standard Fubini - Study Kahler metric (see e.g. [13] ch. 1), with Kahler form

$$(14) \qquad \Omega = \frac{i}{2\pi} \partial \bar{\partial} \log |z|^2; \qquad |z|^2 = \sum_{j=0}^{n} z_j \bar{z}_j.$$

The form (14) belongs to the first Chern class of the hyperplane section bundle on CP^n ([13] ch. 1), and generates $H^2(CP^n, Z)$.

The induced Kahler structure on Q_{n-1} embedded in CP^n as in formula (3) will be denoted also by Ω. It coincides with the invariant Kahler structure on the symmetric space $SO(n+1)/SO(n-1) \times SO(2)$, (see [3]). By functoriality $[\Omega]$ is the first Chern class of the hyperplane section bundle on Q_{n-1}.

In the following we assume that $n > 3$ and write Q instead of Q_{n-1}.

By the Lefschetz hyperplane section theorem ([13] ch. 1) we have

$$(15) \qquad H^2(Q, Z) = H^2(CP^n, Z) = Z$$

moreover, the (class of the) form Ω generates $H^2(Q, Z)$. Also

$$(16) \qquad H^1(Q, \mathcal{O}) = H^2(Q, \mathcal{O}) = 0$$

where $\mathcal{O}$ is the structure sheaf of Q.

Denoting by $\mathcal{O}^*$ the sheaf of nonvanishing holomorphic functions on Q, the exact exponential sequence on Q is:

$$0 \longrightarrow Z \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0.$$

The corresponding exact cohomology sequence is

$$H^1(Q,\mathcal{O}) \longrightarrow H^1(Q;\mathcal{O}^*) \longrightarrow H^2(Q,Z) \longrightarrow H^2(Q,\mathcal{O}). \tag{17}$$

By (16) the extreme right and left terms in (17) vanish, so we have

$$H^1(Q,\mathcal{O}^*) = H^2(Q,Z) = Z. \tag{18}$$

Thus we may identify the group of (equivalence classes of) holomorphic line bundles on Q ($H^1(Q,\mathcal{O}^*)$) with Z. Moreover each holomorphic line boundle L is a tensorial power of the hyperplane section bundle. We denote the k^{th} power by L_k, and by (18) $c_1(L_k) = k\Omega$.

Probably the shortest way to compute the dimensions of the space $H^0(Q, L_k)$ is to use the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{CP^n}(L_k \times L_{-2}) \xrightarrow{\alpha} \mathcal{O}_{CP^n}(L_k) \xrightarrow{r} \mathcal{O}_Q(L_k) \longrightarrow 0 \tag{19}$$

where the mapping α is multiplication of sections of L_{k-2} by the polynomial $\sum_{j=0}^{n} z_j^2$ defining the quadric Q in CP^n, and r is the restriction mapping.

The corresponding cohomology exact sequence begins with

$$\begin{aligned} 0 &\longrightarrow H^0(CP^n, L_{k-2}) \longrightarrow H^0(CP^n, L_k) \longrightarrow H^0(Q, L_k) \longrightarrow \\ &\longrightarrow H^1(CP^n, L_{k-2}). \end{aligned} \tag{20}$$

The last term on (20) is zero by the Kodaira vanishing theorem ([13] ch. 1). Thus we get:

$$\begin{aligned} \dim H^0(Q, L_k) &= \dim H^0(CP^n, L_k) - \dim H^0(CP^n, L_{k-2}) \\ &= \binom{n+k}{n} - \binom{n+k-2}{n} \end{aligned} \tag{21}$$

the last two numbers being the dimensions of the spaces of homogeneous polynomials in $n+1$ variables of degree k and $k-2$ respectively.

We summarise the above in the following.

PROPOSITION. *The group* Pic (Q_{n-1}) *of all holomorphic line bundles on the quadratic* Q_{n-1} *is isomorphic to* $H^2(Q_{n-1}, Z) \cong Z$. *For the bundle* L_k *with* $c_1(L_k) = k\Omega$, $k \in Z^+$ *we have*

$$\dim H^0(Q_{n-1}, L_k) = \frac{2k+n-1}{n-1} \binom{n+k-2}{n-2}$$

The case $k < 0$ *is trivial.*

4. PROOFS OF THE THEOREMS

In order to apply formula (13) we need the first Chern class of Q_{n-1}. If K_Q is the canonical bundle of Q, then $c_1(Q) = -c_1(K_Q)$ and we may apply the adjunction formula ([13] ch. 1), to the smooth hypersurface Q of degree 2 in CP^n. We obtain

$$c_1(Q_{n-1}) = (n-1)\,\Omega. \tag{22}$$

Now we take an arbitrary holomorphic line bundle L_{N-1} on Q_{n-1}. The space $H^0(Q, L_{N-1})$ is not zero for $N = 1, 2, \dots$. We combine formulae (11), (13) and (22) to get

$$\begin{aligned} &c_1(L_{N-1}) = (N-1)\,\Omega = \sqrt{2\epsilon}\,\Omega - \frac{n-1}{2}\,\Omega \quad \text{i.e.} \\ &\sqrt{2\epsilon} = (N-1) + \frac{n-1}{2}\ . \end{aligned} \tag{23}$$

This gives exactly formula (6) for ϵ_N. Formulae (12) and (23) give formula (4). Formulae (5) and (7) for the multiplicity follow directly from the proposition in Sect 3.

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